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ON THE TOTAL TIME SPENT IN RECORDS BY A DISCRETE UNIFORM SEQUENCE

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Abstract

We consider the sum S_d of record values in a sequence of independent random variables that are uniformly distributed on $1, \ldots, d$. This sum can be interpreted as the total amount of time spent in record lifetimes in the standard renewal theoretic setup. We investigate the distributional limit of S_d and some related quantities as $d \to \infty$. Some explicit values are given for $d = 6$, a case that can be interpreted as a simple game of chance.

Keywords: Records; uniform distribution; perpetuities

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1. Introduction and results

Let $(U_{d,n})_{n\in\mathbb{N}}$ be a sequence of independent random variables, all uniformly distributed on the set $\{1, \ldots, d\}$. Here d is an integer that we assume to be greater than 1 to avoid trivialities; for $d = 6$ we obtain the standard model for the repeated toss of a fair die. We call the ith value $U_{d,i}$ in this sequence a *max-record* if $U_{d,i} > U_{d,j}$ for $j = 1, \ldots, i-1$; similarly, $U_{d,i}$ is a *min-record* if $U_{d,i} < U_{d,j}$ for $j = 1, \ldots, i-1$. In both cases we regard the first value $U_{d,1}$ as a record. The central objects of this note are the *record sums*

$$S_d^+ := \sum_{n=1}^{\infty} \Bigl(\prod_{m<n} \mathbf{1}_{\{U_{d,n} > U_{d,m}\}} \Bigr) U_{d,n}, \qquad S_d^- := \sum_{n=1}^{\infty} \Bigl(\prod_{m<n} \mathbf{1}_{\{U_{d,n} < U_{d,m}\}} \Bigr) U_{d,n}.$$

Throughout the paper we use the convention that an empty product has the value 1 and we let $\mathbf{1}_A$ denote the indicator function of the set A. Obviously, there can be at most d records of either kind in the whole sequence, so these sums are always finite.

Our motivation for this investigation was as follows. Perhaps least reputable, but in line with the history of probability, is the application to a simple game of chance: suppose a fair die is thrown repeatedly, the player wins the record total (S_6^+ in the above setup). Here is an actual data set, with '$\triangle$' and '$\triangledown$' denoting the respective max- and min-records and $S_6^+ = 15$:

$$\underset{\triangledown}{\overset{\triangle}{4}} \quad 4 \quad \underset{\triangledown}{2} \quad 2 \quad \overset{\triangle}{5} \quad 3 \quad 4 \quad 3 \quad 2 \quad \underset{\triangledown}{1} \quad 3 \quad 2 \quad 3 \quad 2 \quad 3 \quad 1 \quad \overset{\triangle}{6}$$

Obviously, if the first throw results in '6', then the game ends with the minimal win possible; the maximum of 21 arises if none of the values from 1 to 6 is skipped. What is the fair price for

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this game? Or, alternatively, if the player wins if the record sum is equal to or greater than k: up to which k is this a favourable game? What are the (approximate) answers to these questions if the die has d sides, with d large? For $d = 6$ the exact answers are given in Section 3 below.

Second, we believe that there is an interesting methodological aspect. We could regard the distributions of interest as elements of some space of probability measures and then proceed with (functional) analytic techniques; again, see Section 3 below. Instead we offer a more probabilistic approach via an almost sure construction based on a suitable background stochastic process that drives a whole family of models (here, those for different d).

Third, this note is part of an ongoing project with the aim of understanding records in renewal theory. For records in general we refer the reader to Chapter 4 in Resnick (1987) and the recent book by Arnold *et al.* (1998). In the standard model of renewal theory, nonnegative random variables X_n, $n \in \mathbb{N}$, are regarded as lifetimes of successive pieces of equipment. These are replaced immediately upon failure so that $S_n := \sum_{m=1}^{n} X_m$ is the time of the nth renewal; see Feller (1971) for renewal theory and its applications. Typical questions arising in the renewal-record context are:

(i) What is the probability that the component in use at time t is a record?

(ii) What is the length of the longest lifetime observed up to time t?

(iii) Regarding $(S_n)_{n\in\mathbb{N}}$ as a random partition of the time interval $\mathbb{R}_+$, what is the amount of time spent in records up to time t?

Scheffer (1995) has shown that in the context of (i) a nontrivial limit probability arises for heavy-tailed lifetime distributions, Grübel (1994) obtained a result on the rank of the current lifetime for the finite mean case. The special case of geometric lifetime distributions in (ii) was important for the analysis of von Neumann addition in Grübel and Reimers (2001) (this algorithm is one of the standard topics in computer science curricula and is explained in Scott (1985), for example). For lifetime distributions with finite support and with $t \to \infty$, (iii) leads to the total time spent in records, the question considered here; for Poisson processes similar problems are currently under investigation.

For our analysis of the record sums we will also need the number of max- and min-records,

$$Y_d^+ := \sum_{n=1}^{\infty} \prod_{m<n} \mathbf{1}_{\{U_{d,n} > U_{d,m}\}},$$

$$Y_d^- := \sum_{n=1}^{\infty} \prod_{m<n} \mathbf{1}_{\{U_{d,n} < U_{d,m}\}}.$$

In our first result we obtain a distributional approximation for the Y-sequence with respect to total variation distance. For distributions P, Q on some σ-field $\mathcal{B}$ this distance is defined by

$$d_{\mathrm{TV}}(P, Q) := \sup_{A\in\mathcal{B}} |P(A) - Q(A)|;$$

we have

$$d_{\mathrm{TV}}(P, Q) = \tfrac{1}{2} \sum_{k\in\mathbb{Z}} |P(\{k\}) - Q(\{k\})|$$

if P and Q are concentrated on the set $\mathbb{Z}$ of integers. We write $\mathcal{L}(Z)$ for the distribution of the random variable Z and use $d_{\mathrm{TV}}(X, Y)$ as an abbreviation for $d_{\mathrm{TV}}(\mathcal{L}(X), \mathcal{L}(Y))$. Convergence in the total variation norm is stronger than convergence in distribution: obviously, $d_{\mathrm{TV}}(X_n, X) \to 0$ implies that $\mathrm{P}(X_n \le x) \to \mathrm{P}(X \le x)$ for all $x \in \mathbb{R}$ as $n \to \infty$.

Finally, let $\mathrm{Po}(\lambda)$ denote the Poisson distribution with parameter λ. For future use let $\mathrm{unif}(S)$ be the discrete uniform distribution on the finite set S, $\mathrm{unif}(0,1)$ denotes the uniform distribution on the unit interval and $N(0,1)$ is the standard normal distribution.

Theorem 1. *With* $\lambda_d := \sum_{k=2}^{d} 1/k$ *and* $\mathcal{L}(Z_d) = \mathrm{Po}(\lambda_d)$ *we have*

$$d_{\mathrm{TV}}(Y_d^{\pm}, 1 + Z_d) \le \frac{\pi^2 - 6}{6 \log d/2} \quad \textit{for all } d \ge 2.$$

As a corollary we see that $(Y_d^{\pm} - \log d)/(\log d)^{1/2}$ is asymptotically standard normal, i.e. converges in distribution to $N(0,1)$ as $d \to \infty$. In the proof it is enough to consider Y_d^+ only, as we obviously have $\mathcal{L}(Y_d^+) = \mathcal{L}(Y_d^-)$, which is essentially a consequence of the elementary fact that $\mathcal{L}(X) = \mathrm{unif}(\{1,\dots,d\})$ implies that $\mathcal{L}(d+1-X) = \mathrm{unif}(\{1,\dots,d\})$.

In our next theorem we obtain convergence in distribution for the normalized record sums S_d^-. The limit distribution is the perpetuity associated with $\mathrm{unif}(0,1)$, by which we mean the distribution of $\sum_{n=1}^{\infty} \prod_{k=1}^{n} U_k$ with $(U_k)_{k\in\mathbb{N}}$ a sequence of independent, $\mathrm{unif}(0,1)$-distributed random variables. Perpetuities arise in an astounding variety of situations, see e.g. Goldie and Grübel (1996) and the references given there.

Theorem 2. *Let Z be a random variable whose distribution is the perpetuity associated with* $\mathrm{unif}(0,1)$. *Then*

$$\lim_{d\to\infty} \mathrm{P}\left(\frac{S_d^-}{d} \le x\right) = \mathrm{P}(Z \le x) \quad \textit{for all } x \in \mathbb{R}.$$

Convergence in distribution of a properly rescaled version of S_d^+ is now a simple consequence of these theorems and the fact that the random vectors (Y_d^+, S_d^+) and $(Y_d^-, (d+1)Y_d^- - S_d^-)$ have the same distribution.

Corollary 1. *Let Z be a random variable with distribution* $N(0,1)$. *Then*

$$\lim_{d\to\infty} \mathrm{P}\left(\frac{S_d^+ - d\log d}{d\sqrt{\log d}} \le x\right) = \mathrm{P}(Z \le x) \quad \textit{for all } x \in \mathbb{R}.$$

It also follows from these results that $S_d^+/(dY_d^+)$ converges to 1 in probability, which is remarkable (on first sight) as $S_d^+ \le dY_d^+$: to a first approximation the max-records are all of size d in the sense that the relative proportion of those less than $(1-\varepsilon)d$ tends to 0 for all $\varepsilon > 0$. This can be made more precise: the distributional equality used for Corollary 1 implies that $dY_d^+ - S_d^+$ has the same distribution as $S_d^- - Y_d^-$, hence the above theorems yield the convergence in distribution of $Y_d^+ - d^{-1}S_d^+$. As $Y_d^+/\log d$ converges to 1 in probability we obtain that

$$\lim_{d\to\infty} \mathrm{P}\left(1 - \frac{S_d^+}{dY_d^+} > \frac{\varepsilon}{\log d}\right) = \mathrm{P}(Z > \varepsilon)$$

for all $\varepsilon > 0$, with Z as in Theorem 2.

2. Proofs

We use the background probability space

$$(\Omega_0, \mathcal{A}_0, \mathrm{P}_0) := ((0,1), \mathcal{B}_{(0,1)}, \mathrm{unif}(0,1))^{\otimes \mathbb{N}}$$

together with the random variables

$$U_{0,n} : \Omega_0 \to (0,1), \quad \omega = (\omega_m)_{m\in\mathbb{N}} \mapsto \omega_n \quad \text{for all } n \in \mathbb{N};$$

these are independent and uniformly distributed on the unit interval. The corresponding sequence of min-records is $(U_{0,\tau_n})_{n\in\mathbb{N}}$, where $(\tau_n)_{n\in\mathbb{N}}$ is defined recursively by

$$\tau_1 \equiv 1, \qquad \tau_{n+1} := \inf\{m > \tau_n : U_{0,m} < U_{0,\tau_n}\}.$$

Let $\delta(x)$ denote the unit mass in x. It is known that $\sum_{n=1}^{\infty} \delta(U_{0,\tau_n})$ is a Poisson random measure on the interval $(0,1]$ with intensity ν,

$$\nu((t,1]) = -\log t \quad \text{for } 0 < t \le 1;$$

see e.g. Proposition 4.1(iii) in Resnick (1987) and note that we deal with min-records. Writing $\lceil x \rceil$ for the smallest integer greater than or equal to x we further put

$$U_{d,n} := \lceil dU_{0,n} \rceil \quad \text{for all } d, n \in \mathbb{N};$$

then, for all $d \in \mathbb{N}$, $(U_{d,n})_{n\in\mathbb{N}}$ is a sequence of independent and $\mathrm{unif}(\{1,\dots,d\})$-distributed random variables. A value $k \in \{2,\dots,d\}$ appears as min-record in the sequence $(U_{d,n})_{n\in\mathbb{N}}$ if and only if at least one of the values U_{0,τ_n} falls into the interval $((k-1)/d, k/d]$. The structural result quoted above implies that the random number $V_{d,k}$ of such hits has a Poisson distribution with parameter

$$\lambda_{d,k} := \nu\left(\left(\frac{k-1}{d}, \frac{k}{d}\right]\right) = \log\left(1 + \frac{1}{k-1}\right), \qquad k = 2,\dots,d.$$

Also, $V_{d,2},\dots,V_{d,d}$ are independent. With $W_{d,k} := \mathbf{1}_{\mathbb{N}}(V_{d,k})$ we therefore have

$$Y_d^- = 1 + \sum_{k=2}^{d} W_{d,k},$$

where $W_{d,k}$ has a Bernoulli distribution with parameter

$$p_{d,k} := \mathrm{P}_0(V_{d,k} > 0) = 1 - \exp(-\lambda_{d,k}) = \frac{1}{k}$$

and $W_{d,2},\dots,W_{d,d}$ are independent. (The same representation holds for the number of max-records among the first d of any sequence of independent and identically distributed random variables with continuous distribution function; see Arnold *et al.* (1998, p. 24)). This is the classical situation for Poisson approximation: Barbour *et al.* (1992) give the general result

$$d_{\mathrm{TV}}\left(\mathcal{L}\left(\sum_{k=2}^{d} W_{d,k}\right), \mathrm{Po}\left(\sum_{k=2}^{d} p_{d,k}\right)\right) \le \frac{1}{\sum_{k=2}^{d} p_{d,k}} \sum_{k=2}^{d} p_{d,k}^2.$$

Applying this in the above context and using

$$\sum_{k=2}^{d} p_{d,k} = \sum_{k=2}^{d} \frac{1}{k} \geq \log d - \log 2, \qquad \sum_{k=2}^{d} p_{d,k}^2 = \sum_{k=2}^{d} \frac{1}{k^2} \leq \frac{\pi^2}{6} - 1,$$

we obtain the statement of Theorem 1.

To prove Theorem 2 we first identify the distribution of $Z := \sum_{n=1}^{\infty} U_{0,\tau_n}$, the sum of all min-records in the sequence $(U_{0,n})_{n\in\mathbb{N}}$. Applying the transformation $t \mapsto -\log t$ to the points of the Poisson random measure introduced above, we obtain a Poisson process on $[0, \infty)$ with constant intensity 1, which means that the position of the first point $V_1 := -\log U_{0,\tau_1}$ and the spacings

$$V_n := -\log U_{0,\tau_n} + \log U_{0,\tau_{n-1}}, \qquad n > 1,$$

are independent and exponentially distributed with parameter 1. From

$$Z = \sum_{n=1}^{\infty} \exp\biggl(-\sum_{m=1}^{n} V_m\biggr) = \sum_{n=1}^{\infty} \prod_{m=1}^{n} \tilde{U}_m,$$

with $\tilde{U}_m := \exp(-V_m)$, we now obtain the desired representation as the variables $\tilde{U}_m,\ m \in \mathbb{N}$, are independent and unif$(0,1)$-distributed.

We next obtain a lower bound for S_d^-/d within the above construction. Let $N \in \mathbb{N}$ be given. For any d with

$$\frac{1}{d} < \min\{U_{0,\tau_{n-1}} - U_{0,\tau_n} : 1 < n \leq N\},$$

the values $\lceil dU_{0,\tau_m}\rceil,\ m = 1,\dots,N$, are all different so that $S_d^-/d \geq \sum_{n=1}^{N} U_{0,\tau_n}$. Letting first $d \to \infty$ and then $N \to \infty$ in this inequality we obtain

$$\liminf_{d\to\infty} \frac{1}{d} S_d^- \geq Z \quad \text{almost surely.}$$

For a corresponding upper bound we first note that the discretization resulting in S_d^-/d adds at most $1/d$ to each of the contributing U_{0,τ_n}-values, hence

$$\frac{1}{d} S_d^- \leq Z + \frac{1}{d} Y_d^-.$$

From the proof of Theorem 1 we know that $Y_d^- - 1 = \sum_{k=2}^{d} W_{d,k}$ with $W_{d,2},\dots,W_{d,d}$ independent and $\mathrm{P}(W_{d,k} = 1) = 1 - \mathrm{P}(W_{d,k} = 0) = 1/k$. Using Markov's inequality we obtain

$$\mathrm{P}\biggl(\frac{1}{d} Y_d^- > \varepsilon\biggr) \leq \frac{1}{\varepsilon d}\, \mathrm{E}\, Y_d^- = \frac{1}{\varepsilon d} \sum_{k=1}^{d} \frac{1}{k}.$$

This shows that Y_d^-/d converges to 0 in probability as $d \to \infty$, hence it follows from the upper and lower bound that S_d^-/d converges in distribution to Z.

3. Comments and examples

An approach based on analytic methodology to the problems considered in this paper might begin with deriving a recursion relation for the probabilities of interest. These lead to equations for some associated generating functions and thereby open up the connection to a wealth of analytic techniques. Recursion relations can also be used to compute the distributions of interest for finite d. Suppose that we want to calculate the joint distribution of Y_d^- and S_d^-, i.e. the values

$$\alpha(d,i,j) := \mathrm{P}(Y_d^- = i, S_d^- = j).$$

We consider the following models,

$$(\Omega_d, \mathcal{A}_d, \mathrm{P}_d) := (\{1,\dots,d\}, \mathcal{P}(\{1,\dots,d\}), \mathrm{unif}(\{1,\dots,d\}))^{\otimes\mathbb{N}},$$
$$U_{d,n} : \Omega_d \to \{1,\dots,d\}, \quad \omega = (\omega_m)_{m\in\mathbb{N}} \mapsto \omega_n \quad \text{for all } n \in \mathbb{N},$$

and construct a family of transformations $T_d : \Omega_d \to \Omega_{d-1}$ relating these probability spaces by

$$(T_d(\omega))_n := U_{d,\tau_n(\omega)}(\omega), \qquad n \in \mathbb{N},$$

where the sequence $(\tau_n)_{n\in\mathbb{N}_0}$ is defined recursively by

$$\tau_0 \equiv 0, \qquad \tau_{n+1} := \inf\{m > \tau_n : U_{d,m} < d\}.$$

In words: $T_d(\omega)$ is obtained from ω by deleting all d-values. In these definitions we follow the convention that $\inf\varnothing = \infty$ and we put $U_{d,\infty} \equiv 0$. With respect to P_d and the other distributions on Ω_d that we introduce below, $\{\tau_n = \infty\}$ only happens on a null set and is therefore without relevance for the probabilities $\alpha(d,i,j)$. It is easy to check that the image of P_d under T_d is P_{d-1} and that the same image measure arises if we replace P_d by the conditional probability measures $\mathrm{P}_d(\cdot \mid U_{d,1} < d)$ and $\mathrm{P}_d(\cdot \mid U_{d,1} = d)$.

The pair (Y_d^-, S_d^-) can be regarded as a function on Ω_d, with values in $\mathbb{N}\times\mathbb{N}$. On $\{U_{d,1} < d\}$ we obviously have $(Y_d^-, S_d^-) = (Y_{d-1}^-, S_{d-1}^-) \circ T_d$, and on $\{U_{d,1} = d\}$ we have $(Y_d^-, S_d^-) = (Y_{d-1}^-, S_{d-1}^-) \circ T_d + (1, d)$. Combining these considerations we arrive at

$$\begin{aligned}\alpha(d,i,j) &= \mathrm{P}_d(Y_d^- = i, S_d^- = j \mid U_{d,1} < d)\,\mathrm{P}_d(U_{d,1} < d) \\ &\quad + \mathrm{P}_d(Y_d^- = i, S_d^- = j \mid U_{d,1} = d)\,\mathrm{P}_d(U_{d,1} = d) \\ &= \Big(1 - \frac{1}{d}\Big)\alpha(d-1,i,j) + \frac{1}{d}\alpha(d-1,i-1,j-d),\end{aligned}$$

a relation that the experienced coin-tosser accepts without the above formal ado. Together with the obvious fact that $(Y_1^-, S_1^-) = (1,1)$ this can now be used to calculate the joint probability mass function (PMF) of Y_d^- and S_d^- and from

$$\mathcal{L}((Y_d^+, S_d^+)) = \mathcal{L}((Y_d^-, (d+1)Y_d^- - S_d^-))$$

we then also get the joint PMF of Y_d^+ and S_d^+. Table 1 gives the latter values for $d = 6$, zero entries are left blank. We have $\mathrm{E}\, S_6^+ = 8028/720 = 11.15$ and $\mathrm{P}(S_6^+ \geq k)$ drops from $432/720 = 0.6$ to $290/720 \approx 0.4028$ as k increases from 11 to 12. The last line contains the (rounded) probabilities that are obtained from the shifted Poisson approximation in Theorem 1.

The approximation is apparently not very precise for $d = 6$. It might be worth noting in this context that the results in Chapter 2 of Barbour *et al.* (1992) can be used to obtain a lower

TABLE 1: Joint distribution of Y_6^+ and S_6^+ (all probabilities are multiplied by 720).

	Y						
S	1	2	3	4	5	6	PMF
6	120						120
7		24					24
8		30					30
9		40	6				46
10		60	8				68
11		120	22				142
12			39	2			41
13			50	3			53
14			40	10			50
15			60	13			73
16				22	1		23
17				15	2		17
18				20	3		23
19					4		4
20					5		5
21						1	1
PMF	120	274	225	85	15	1	720
Approx.	169	245	178	86	31	9	717

bound of the order $(\log d)^{-1}(\log \log d)^{-2}$ for the total variation distance, hence the rate of convergence is indeed very slow.

At present we do not see how to use the above recursion to derive limit results of the type given in Section 1. However, it is possible to consider the distributions as a whole rather than the individual probabilities and this indeed leads to an alternative approach. This has been carried out in detail by Reimers (2000), based on the ideas of Rösler's (1991) analysis of the Quicksort algorithm. In this approach, a recursive relation for the distributions μ_d, $d \in \mathbb{N}$, of interest is first established. After a suitable rescaling the recursion can be regarded as a transformation T on some complete metric space $(\mathbb{M}, \rho)$ of probability measures. It is then shown that the limit distribution has to be a fixed point of T and that T is a contraction; Banach's fixed point theorem can then be applied. The details can be rather technical as the transformation may depend on d; also, a tightness argument in order to establish the existence of a limit point for the sequence of rescaled μ_ds might be needed.

Reimers (2000) used this method for a proof of Theorem 2, with μ_d the distribution of $W_d := (S_d^- - \mathrm{E}\, S_d^-)/d$ (this already incorporates the rescaling). The space $\mathbb{M}$ consists of the probability measures on the real line with finite second and vanishing first moment, ρ is a Wasserstein metric as in Rösler (1991). The starting point for obtaining a recursive relation is a decomposition with respect to the first value in the underlying sequence of 'dice throws' which shows that S_d^- is equal in distribution to the sum $U_d + S_{U_d-1}^-$, with $\mathcal{L}(U_d) = \mathrm{unif}(\{1, \ldots, d\})$ and $U_d, S_1^-, \ldots, S_{d-1}^-$ independent. For the rescaled variables W_d this implies that

$$\mathcal{L}(W_d) = \mathcal{L}\left(\frac{U_d - 1}{d} W_{U_d-1} + C_d(U_d)\right) \quad \text{with } C_d(k) := \frac{k + \mathrm{E}\, S_{k-1}^- - \mathrm{E}\, S_d^-}{d}.$$

A separate argument shows that $\lim_{d\to\infty} \mathrm{E}\, S_d^-/d = 1$, which leads to $C_d(U_d) \approx 2U - 1$ for d large with $\mathcal{L}(U) = \mathrm{unif}(0,1)$. Hence, if W_d converges to some W_∞ as $d \to \infty$ in some sense that implies convergence in distribution and convergence of the respective first moments, then we would expect that

$$\mathcal{L}(W_\infty) = \mathcal{L}(U W_\infty + C_\infty(U)) \quad \text{with } C_\infty(x) := 2x - 1,$$

which means that $\mathcal{L}(W_\infty)$ is a fixed point of the transformation

$$T : \mathbb{M} \to \mathbb{M}; \quad \mu \mapsto \mathcal{L}(UX + 2U - 1)$$

with X, U independent, $\mathcal{L}(X) = \mu$ and $\mathcal{L}(U) = \mathrm{unif}(0,1)$. This transformation is known to be closely related to perpetuities; see Goldie and Grübel (1996).

In conclusion it seems that in the situation considered here the probabilistic approach, based on the construction of a suitably rich background object, offers some advantages in terms of the general understanding of the random mechanism; the authors had a similar experience in connection with the analysis of von Neumann addition in Grübel and Reimers (2001). To give a specific *nonasymptotic* example in the context of the present paper, consider the events A_k that the value k arises in the max-record sequence, $k = 1, \ldots, d$: from the constructions in Section 2 it is quite obvious that these events are independent. On the other hand the identification of the limit distribution as the fixed point of some transformation can be very useful if numerical approximations are required, the perpetuity appearing in our results providing a case in point.

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